STATIC LOWER BOUNDS FOR GAUSSIAN RISKS AND COMONOTONIC ASSET PRICES ON ARBITRAGE-FREE MARKET

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ABSTRACT

Consider a Gaussian random vector $\underline{X}=(X_1,\ldots X_n)$ of which we can only observe the marginal distributions. Our goal is to construct an optimal static lower bound for $g_B(\underline{X},K)=(S-K)_+$ where $S=X_1+\ldots+X_n$, in terms of random variables (*rv's*) $g_C(X_i,K_i)=(X_i-K_i)_+$ and $g_P(X_i,K_i)=(K_i-X_i)_+$, respectively. We interpret X_i's as financial or actuarial risks. We mention the paper of Hobson et al. (2005) who studied a lower bound for basket options of two components, which is a special case of our problem for nonnegative rv's X_i's and n=2. A related problem of *upper* bound for basket options is investigated, e.g., in Rüschendorf (2005), p.34, where *comonotonic* joint distributions of asset prices are involved. Below we present the main our result.

Let μ_i *and* σ_i^2 *be the fixed values of mean and variance of* X_i, $i=1,\ldots,n$, σ_m^2 *be the largest of those variances,* $\sigma=(\sigma_m-\sum_{i\neq m}\sigma_i)_+$ *and* $M=\mu_1+\ldots+\mu_n$. *If* $\sigma>0$ *then for all nonrandom vectors* $\underline{x}$, *it holds:*

$$g_B(\underline{x},K)\geq g_C(x_m,K_m)-\sum_{i\neq m}g_P(x_i,K_i)=:g_{SR}^{+}(\underline{x})\ ,\quad \text{moreover}\quad \min \mathbf{E}g_B(\underline{X},K)=\mathbf{E}g_B(\underline{X}^{*},K)=\mathbf{E}g_{SR}^{+}(\underline{X}^{*})\ .$$

Hereafter minimum is taken over all possible Gaussian vectors $\underline{X}$ *with fixed marginal distributions, and* $\underline{X}^{*}=(-\sigma_1\gamma,-\sigma_2\gamma,\ldots,+\sigma_m\gamma,-\sigma_{m+1}\gamma,\ldots,-\sigma_n\gamma)$ *with* $\gamma\sim N(0,1)$, *and* $K_m=\sigma_m\sigma^{-1}(K-M)+\mu_m$, $K_i=-\sigma_i\sigma^{-1}(K-M)+\mu_i$, $i\neq m$.

If $\sigma=0$ *and* $M\leq K$ *then* $g_B(\underline{x},K)\geq 0$ *and* $\min \mathbf{E}g_B(\underline{X},K)=0$. *Thus, in this case an optimal lower bound is equal to zero. Finally, if* $\sigma=0$ *and* $M>K$ *then for any* Δ *and all nonrandom* $\underline{x}$, *it holds:*

$$g_B(\underline{x},K)\geq g_C(x_m,K+\Delta)-\sum_{i\neq m}g_P(x_i,-\Delta(n-1)^{-1})=:g_{SR}^{0}(\underline{x},\Delta)\ ,\quad \text{moreover}\quad \min \mathbf{E}g_B(\underline{X},K)=M-K=\lim_{\Delta\to+\infty}\mathbf{E}g_{SR}^{0}(\underline{X},\Delta)\ .$$

Therefore, in this case the rv's $g_{SR}^{0}(\underline{x},\Delta)$ *with* $\Delta>0$ *provide the asymptotically optimal lower bounds.*

Also, for an arbitrage-free market with one underlying asset, we show that subsequent values of the asset price form a comonotonic random vector only under a deterministic linear relationship (see [1] for the definition of comonotonic vector).
Keywords: Gaussian risks, Optimal lower bounds, Basket options, Arbitrage-free market, Comonotonic random vector, Linear relationship.

References

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